IJBPAS, July, 2015, 4(7): 448-453

ISSN: 2277–4998

International Journal of Biology, Pharmacy and Allied Sciences (IJBPAS)

'A Bridge Between Laboratory and Reader'

www.ijbpas.com

A QUANTUM MECHANICAL PARTICLE IN CURVED SPACE-TIME

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ABSTRACT

In this paper exact solutions of the general relativistic Klein-Gordon equation areobtained for curved space time. In fact our solutions are in the Schwarzschild metric with considering t=0 and $\theta = \frac{\pi}{2}$. This paper indicates connection between gravity and quantum mechanics thus our work constitute another strong motivation to the study of the Klein-Gordon equation in curved space.

PACS numbers: 03. 65. Pm, 04. 60. - m, 98. 80. Cq.

Key words: Klein-Gordon equation, curved space time, Schwarzschild metric.

INTRODUCTION

The general method to consider the interaction between relativistic quantum mechanical particles and gravity is to evaluate their wave equations. The general relativistic form of these wave equations explains the interaction between these particles and gravitation.These relativistic forms may not be significant at the atomic scale because of the weakness of the gravitational effects, but the physics governing these particles plays an important role in astrophysics and cosmology due to the dominant roles of gravitational effects, like particle creation by black holes which has an importance for understanding early universe. In addition to that, studying single particle states is very important to be able to construct a unified theory of gravitation and quantum mechanics.In this paper we consider the general relativistic Klein-Gordon equation for quantum mechanical particles of spin-zero in the Schwarzschild metric with considering t=0 and $\theta = \frac{\pi}{2}$, this metric represent the static spherically symmetric gravitational field in the empty space surrounding some massive spherical object such as a star and use the Nikiforov-Uvarov method[1]. Also we can conclude physics of a particle in the vicinity of a spherical object of mass M, in particular the trajectories of freely falling massive particles. Furthermore this metric is valid down to the surface of the object, if the surface of the massive body contracts within the Schwarzschild radius ($r = 2\mu$), then the objects becomes a Schwarzschild black hole. In this manuscript we will attention to the region $r \succ 2\mu$, where $\mu = \frac{GM}{c^2}$.In recent years, investigation of general relativistic equation in gravitational fields has been at the center of interest hence we can cited some of these papers as [2-6]. These works indicate connection between gravity and quantum mechanics thus our work constitute another strong motivation to the study of the Klein-Gordon equation in curved space.

1. A spin-zero particle in Schwarzschild space-time

The Klein-Gordon equation in the curved space-time is[7],

$$(\Box + \frac{m^2c^2}{\hbar^2})\Phi(x) \equiv (g^{\mu\nu}\partial_\mu\partial_\nu - g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu}\partial_\lambda + \frac{m^2c^2}{\hbar^2})\Phi(x) = 0, \qquad (1)$$

The Schwarzschild line element introduced as below,

$$ds^2 = c^2(1-\frac{2\mu}{r})dt^2 - (1-\frac{2\mu}{r})^{-1}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta\, d\phi^2, \qquad (2)$$

We suppose that $t = 0$, $\theta = \frac{\pi}{2}$, so we have,

$$ds^2 = -(1-\frac{2\mu}{r})^{-1}dr^2 - r^2\, d\phi^2, \qquad (3)$$

For solving the general relativistic Klein-Gordon equation, we need to obtain the geodesic equation then determine the connection coefficient as,

$$(1-\frac{2\mu}{r})^{-1}\ddot{r}-(1-\frac{2\mu}{r})^{-2}\frac{\mu}{r^2}\dot{r}^2-r\dot{\phi}^2=0, \quad (4)$$

By simplify the above equation, we have

$$\ddot{r}-(1-\frac{2\mu}{r})^{-1}\frac{\mu}{r^2}\dot{r}^2-(1-\frac{2\mu}{r})r\dot{\phi}^2=0, \quad (5)$$

So we find the non-zero connection coefficient as follow,

$$\Gamma^r_{rr}=-(1-\frac{2\mu}{r})^{-1}\frac{\mu}{r^2}, \quad \Gamma^r_{\phi\phi}=-(1-\frac{2\mu}{r})r, \quad (6)$$

So we have,

$$\{g^{rr}\partial_r^2-g^{rr}\Gamma^r_{rr}\partial_r+g^{\phi\phi}\partial_\phi^2-g^{\phi\phi}\Gamma^r_{\phi\phi}\partial_r+\frac{m^2c^2}{\hbar^2}\}\Phi_{n,\lambda}(r,\phi)=0, \quad (7)$$

We now substitute these connection coefficient into the expression (7) in order to obtain the general relativistic Klein-Gordon equation as,

$$\{\partial_r^2+(\frac{\mu}{r^2(1-\frac{2\mu}{r})}+\frac{1}{r})\partial_r-\frac{m^2c^2}{\hbar^2(1-\frac{2\mu}{r})}+\frac{1}{r^2(1-\frac{2\mu}{r})}\partial_\phi^2\}\Phi_{n,\lambda}(r,\phi)=0, \quad (8)$$

Therefore we have

$$\partial_\phi=\frac{i}{\hbar}L_z=i\lambda, \qquad \partial_\phi^2=-\lambda^2$$

Thus, we can rewrite the Equation (8) as follow,

$$\{\partial_r^2+(\frac{\mu}{r^2(1-\frac{2\mu}{r})}+\frac{1}{r})\partial_r-\frac{m^2c^2}{\hbar^2(1-\frac{2\mu}{r})}+\frac{-\lambda^2}{r^2(1-\frac{2\mu}{r})}\}\Phi_{n,\lambda}(r)=0, \quad (9)$$

In this case, we choose $S=\frac{1}{r}, m^2=s^2m_0^2$, so the Eq. (9) come to Eq. (10)

$$\frac{d^2\Phi}{ds^2}+\frac{-\mu}{s(1-2\mu s)}\frac{d\Phi}{ds}+\frac{1}{s^2(1-2\mu s)^2}[\frac{-m_0^2c^2}{\hbar^2}+\frac{2\mu m_0^2c^2}{\hbar^2}s-\lambda^2+2\mu\lambda^2 s]\Phi_{n,\lambda}(r)=0, \qquad (10)$$

Form of the Eq. (10) is similar to the Nikiforov-Uvarov second order differential equations. So this method enables us to solve this equation.

2. Exact solutions of the problem

The second order differential equation is,

$$\frac{d^2}{ds^2}\psi_n(s)+\frac{\alpha_1-\alpha_2 s}{s(1-\alpha_3 s)}\frac{d}{ds}\psi_n(s)+\frac{-\xi_1 s^2+\xi_2 s-\xi_3}{[s(1-\alpha_3 s)]^2}\psi_n(s)=0. \qquad (11)$$

Furthermore we have

$$\alpha_4=\frac{1}{2}(1-\alpha_1), \qquad (12)$$

$$\alpha_5=\frac{1}{2}(\alpha_2-2\alpha_3), \qquad (13)$$

$$\alpha_6=\alpha_5^2+\xi_1, \qquad (14)$$

$$\alpha_7=2\alpha_4\alpha_5-\xi_2, \qquad (15)$$

$$\alpha_8=\alpha_4^2+\xi_3, \qquad (16)$$

$$\alpha_9=\alpha_3\alpha_7+\alpha_3^2\alpha_8+\alpha_6, \qquad (17)$$

The energy eigenvalues equation can be readily obtained with above results as follow,

$$\alpha_2 n-(2n+1)\alpha_5+(2n+1)(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8})+n(n-1)\alpha_3+\alpha_7+2\alpha_3\alpha_8+2\sqrt{\alpha_8\alpha_9}=0. \qquad (18)$$

In order to obtain the wave functions, one can use the following relations,

$$\alpha_{10}=\alpha_1+2\alpha_4+2\sqrt{\alpha_8}, \qquad (19)$$

$$\alpha_{11}=\alpha_2-2\alpha_5+2(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}), \qquad (20)$$

$$\alpha_{12}=\alpha_4+\sqrt{\alpha_8}, \qquad (21)$$

$$\alpha_{13}=\alpha_5-(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}). \qquad (22)$$

And the wave functions can be written as,

$$\Psi_n(s)=s^{\alpha_{12}}(1-\alpha_3 s)^{-\alpha_{12}-(\alpha_{13}/\alpha_3)}P_n^{(\alpha_{10}-1,(\alpha_{11}/\alpha_3)-\alpha_{10}-1)}(1-2\alpha_3 s),$$

In order to use this method , we can derive the eigenvalues relation as follow,

$$2\mu n(1+n)+2\mu\frac{m_0^2c^2}{\hbar^2}+2\mu\lambda^2+\mu(1+\mu^2)+(2n+1)\mu\sqrt{\mu^2-4\mu+1}+$$
$$2(2n+1)\mu\sqrt{\lambda^2+\frac{1}{4}(1+\mu)^2+\frac{m_0^2c^2}{\hbar^2}}=0, \qquad (23)$$

Hence we can obtain the wave function as follow,

$$\Phi_{n,\lambda}(s)=S^{\frac{1+\mu}{2}+\sqrt{\lambda^2+\frac{1}{4}(1+\mu)^2+\frac{m_0^2c^2}{\hbar^2}}}(1-2\mu S)^{\frac{1-\mu}{2}+\frac{\sqrt{\mu^2-4\mu+1}}{2}}P_n^{(2\sqrt{\lambda^2+\frac{1}{4}(1+\mu)^2+\frac{m_0^2c^2}{\hbar^2}},\sqrt{\mu^2-4\mu+1})}(1-4\mu S), \qquad (24)$$

$$\Phi_{n,\lambda}(r)=(\frac{1}{r})^{\frac{1+\mu}{2}+\sqrt{\lambda^2+\frac{1}{4}(1+\mu)^2+\frac{m_0^2c^2}{\hbar^2}}}(1-\frac{2\mu}{r})^{\frac{1-\mu}{2}+\frac{\sqrt{\mu^2-4\mu+1}}{2}}P_n^{(2\sqrt{\lambda^2+\frac{1}{4}(1+\mu)^2+\frac{m_0^2c^2}{\hbar^2}},\sqrt{\mu^2-4\mu+1})}(1-\frac{4\mu}{r}).$$

Where the Jacobi polynomial is,

$$P_n^{(c,d)}(z)=2^{-n}\sum_{p=0}^{n}\binom{n+c}{P}\binom{n+d}{n-p}(1-z)^{n-p}\ (1+z)^p$$

$$P_n^{(c,d)}(z)=\frac{\Gamma(n+c+1)}{n!\Gamma(n+c+d+1)}\sum_{r=0}^{n}\binom{n}{r}\frac{\Gamma(n+c+d+r+1)}{\Gamma(r+c+1)}(\frac{z-1}{2})^r, \qquad (25)$$

CONCLUSION

In this paper we have considered the interaction between relativistic quantum mechanical particles and gravity thus we have studied the general relativistic Klein-Gordon equation in the Schwarzschild metric when $t=0,\ \theta=\frac{\pi}{2}$. In order to use this equation we obtain wave functions andthe eigenvalues relation.

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